

A LOWER BOUND ON INSPECTION TIME FOR COMPLEX SYSTEMS WITH WEIBULL TRANSITIONS

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Abstract: The paper studies the expectation of the inspection time in complex aging systems. Under reasonable assumptions, this problem is equivalent to studying the expectation of the length of the shortest path in the directed degradation graph of the systems where the parameters are obtained from experts. The expectation itself being sometimes out of reach, in closed form or even through Monte Carlo simulations in the case of large systems, we study the bound of Dyer, Frieze and McDiarmid which provides an interesting upper bound in the case of exponential transition times between degradation states. On the other hand, we show that this bound does not hold for Weibull distributions. Another problem is that lower bounds are much more useful in the context of estimating inspection times before failure. Such a rigourous lower bound is presented for the case of Weibull distribution with reasonable values of the shape parameter.

1 Introduction

1.1 Problem statement

Consider a complex system whose n degradation states have been identified by experts. Let node 1 represent the state where the system is considered as new and let node n be the state of unacceptable degradation. All maximum paths from any node of the graph end at node n as in the figure below. The system is supposed to possibly evolve from a degradation state to any neighbor in the corresponding directed graph. The transition time between any two given states is assumed to follow a Weibull distribution whose parameters are given by experts or are estimated if the number of observations is sufficiently large. Using Bayesian statistics both informations can also be merged.

Assume we start with a brand new system. Then, evolution of the system starts in state 1. Maintenance policies require that the system be inspected before reaching state n, i.e. unacceptable degradation. Such examples of complex systems have been studied in [1]. The problem posed in this paper is to provide a lower bound on acceptable inspection times.

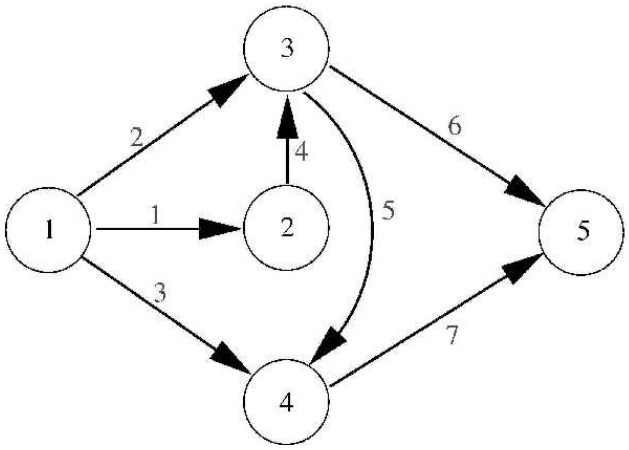

Figure 1: A simple graph

1.2 Inspection times and shortest paths

In order to simplify the analysis, we assume that evolution inside the degradation graph proceeds following the rule that starting from one node i, the system goes to state j minimizing the transition time among neighbors of state i. Therefore, acceptable inspection times will be the times lower than *the shortest path* from state 1 to state n where each edge is weighted by its transition time. In general situations, we thus may ask for

- an estimator of the expected length of the shortest path from 1 to n,
- a confidence interval for the expected time. path.

This task is in general impossible to achieve because of the hudge number of observations this should require in practice. The goal of this paper is to propose a lower bound on the expected length of the shortest path. On the other hand, approximate confidence intervals seem very difficult to obtain. A possible way of doing this may be the use of Talagrand's inequalities but this issue will not be discussed here.

2 The Dyer-Frieze-McDiarmid inequality for exponential transitions

An important step in the search of good bounds for expectations in combinatorial problems was achieved by Dyer, Frieze and Mc Diarmid in [2]; see also [3]. Their bound is an upper bound to the expectation of the optimal value. In comparison, the main objective of our work is to obtain a lower bound but using Dyer-Frieze-McDiarmid's bound gives a first understanding of the problem.

2.1 Linear programming formulation

The main idea is to convert the problem into an equivalent linear programming problem, when possible. Many combinatorial optimization problems cannot be transformed in this manner but it is well known that this is the case for the shortest path problem. Consider the following *extended incidence matrix* A of the oriented degradation graph. Its rows are indexed by the nodes of the graph while its columns are indexed by its edges with an extra column of all ones. In each column indexed by edge (i, j), set the ith component to -1, the jth component to 1 and set all other entries to zero. For instance, the extended incidence matrix for the graph of figure **??** is given by

$$A = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Now, the fact that we start at node 1 and end at node n is encoded in the vector $b = [-1, 0, 0, 0, 1]^t$. Then, the shortest path problem is equivalent to the linear program

$$\begin{aligned} z^\star = \quad & \min \quad c^T x \\ & \text{s.t.} \quad Ax = b \\ & \qquad\quad x \geq 0 \end{aligned} \tag{1}$$

where the random vector c contains the transition times for each edge of the graph.

2.2 The case of exponential transition times

We now apply the Dyer-Frieze-McDiarmid bound to our shortest path problem in the case where the transition times are independent and exponential. In this case, the mean residual times, i.e. the variables $E[c_i \mid c_i \geq h]$ satisfy the equality

$$E[c_i \mid c_i \geq h] = E[c_i] + h.$$

Then, an *upper bound* on the expected length of the shortest path is given by the following theorem.

Theorem 1 *(Dyer-Frieze-McDiarmid Inequality [2]) Assume that the random costs c_i are independent and satisfy*

$$E[c_i \mid c_i \geq h] \geq E[c_i] + \alpha h$$

for some $\alpha \in (0, 1]$. Then for any matrix $A \in \mathbb{R}^{n \times m}$ and any vector $b \in \mathbb{R}^n$, the optimal value z^* of the general linear program (1) satisfies

$$E[z^*] \leq \max_{S\,:\,\#S=n} \sum_{i \in S} E[c_i] x_i \tag{2}$$

for any feasible solution x, i.e. any x satisfying $Ax = b$.

A direct application of this theorem gives the following interesting corollary.

Corollary 1 *Consider problem (1) where the random costs are assumed to be independent and exponentially distributed and consider the associated deterministic linear program*

$$\begin{aligned} \zeta^\star = \quad & \min \quad E[c]^T x \\ & s.t. \quad Ax = b \\ & \qquad\quad x \geq 0 \end{aligned} \tag{3}$$

where the random costs are replaced by their expected values. Then the shortest path problem with optimal value denoted by z^* satisfies

$$E[z^*] \leq \zeta^*.$$

Proof. Take x equal to any binary vector minimizing (3). It is clear that the number of ones in this vector is less than the number of nodes in the graph. Then, the maximum value over all sets S of cardinality n in the right hand term in (2) is obtained when S is taken to be the set of indices i for which $x_i = 1$. Thus $\sum_{i \in S} E[c_i]x_i$ is exactly the cost of x , i.e. ζ^*. □

An important conclusion of this corollary is that, contrarily to the intuitive idea prescribed by common practice, replacing the random costs by their expected values cannot help for the problem of finding a lower bound to the inspection time of the system. A rigourous lower bound will be presented below.

3 A lower bound on the shortest path and inspection times

3.1 The mean residual time to failure for Weibull distributions

Analyzing the proof of the Dyer-Frieze-McDiarmid bound reveals the importance of studying the mean residual transition times. Here, we recall some results for mean residual times in the case where the variables have a Weibull distribution. Let X be a random variable with Weibull distribution

$$f_X(t) = \frac{\beta}{\eta}\Big(\frac{t}{\eta}\Big)^\beta e^{-(\frac{t}{\eta})^\beta}.$$

Then, the mean residual time to failure (MRTF) is given by

$$G_X(h) \triangleq E[X \mid X \geq h] = \eta e^{(\frac{h}{\eta})^\beta}\Gamma(1 + \frac{1}{\beta}, (\frac{h}{\eta})^\beta),$$

where $\Gamma(a,h)$ is the incomplete gamma function defined by

$$\Gamma(a,h) = \int_h^{+\infty} t^{a-1}e^{-t}dt.$$

Lemma 1 *The first two derivatives of the MRTF for a Weibull distributed variable X are given by*

$$G'_X(h) = \frac{\beta}{\eta^\beta}h^{(\beta-1)}\Big\{\eta e^{(\frac{h}{\eta})^\beta}\Gamma\big(1+\frac{1}{\beta},(\frac{h}{\eta})^\beta\big) - h\Big\}$$

and

$$\begin{aligned} G''_X(h) &= -\frac{\beta^2}{\eta^\beta}h^{(\beta-1)}\Big(1+\beta(\frac{h}{\eta})^\beta\Big) \\ &+ \frac{\beta}{\eta^{(\beta-1)}}h^{(\beta-2)}e^{(\frac{h}{\beta})^\beta}\Gamma(1+\frac{1}{\beta},(\frac{h}{\eta})^\beta)\Big(\beta(\frac{h}{\eta})^\beta+\beta-1\Big). \end{aligned}$$

Moreover, when $\beta \geq 1$ we have

$$\lim_{h\to 0} G'_X(h) = 0 \text{ and } \lim_{h\to+\infty} G'_X(h) = 1$$

and if moreover $\beta < 2$ we have

$$G''_X(h) > 0, \quad \lim_{h\to 0} G''_X(h) = +\infty \text{ and } \lim_{h\to+\infty} G'_X(h) = 0.$$

Using this lemma, we can draw the following simple conclusion.

Proposition 1 *Assume that X is Weibull distributed with $\beta \in (1,2)$. Then, for all $h \geq 0$, we have*

$$E[X \mid X \geq h] \leq E[X] + h.$$

This result will be instrumental in the derivation of the bound.

3.2 A lower bound to the inspection time

In order to derive the lower bound, we need to clarify the behavior of the linear program (1) when subject to random costs. The presentation follows [2] and [3, Chapter 4]. For any family of n different columns of A indexed by B which are linearly independent and such that $A_B^{-1}b \geq 0$, we can find a feasible solution x satisfying $Ax = b$ and $x \geq 0$ in the following way :

$$\begin{cases} x_i = 0 \text{ forall } i \notin B \text{ and} \\ x_B = A_B^{-1}b \end{cases} \tag{4}$$

where x_B is the vector whose components are those of x with indices in B and A_B is the matrix whose columns are those of A whose indices are in B also. We will also sometimes use the notation $N = \{1,\ldots,n\} \setminus B$.

We then have the following standard result in linear programming theory.

Lemma 2 *([2, 3]) Let B be a subset of n indices such that $A_B^{-1}b \geq 0$. A vector x defined by (4) with respect to B will be a solution to program (1) if and only if $c_i \geq c_B^t A_B^{-1} a_i$ for all $i \notin B$ where a_i is the ith column of A.*

A subset of indices B such that the columns of A_B are linearly independent and $A_B^{-1}b \geq 0$ is called a basis of the linear program. Solving the linear program thus consists of finding an optimal basis characterized by the property

$$c_i \geq c_B A_B^{-1} a_i \text{ for all } i \notin B. \tag{5}$$

The reason for appending the extra column of all ones to the incidence matrix is that the standard incidence matrix has rank equals to $n-1$ when the graph is connected. Thus, any basis for the shortest path problem must contain this extra column of ones which is easily seen to be orthogonal to any of the other columns.

Using these definitions, we now present the lower bound to the inspection time in the following theorem.

Theorem 2 *Consider the shortest path problem whose linear programming formulation is given by (1) where the components c_i of the cost vector c are independent and follow a Weibull distribution with parameters η_i and β_i, $i = 1, \ldots, n$. Let x be any feasible solution of (1), i.e. satisfying $Ax = b$ and $x \geq 0$. Then, the expectation of the random optimal value z^* admits the following lower bound*

$$\sum_r p_r \sum_{i \in B_r} E[c_i] x_i \leq E[z^*],$$

where (B_r) is the family of all bases for program (1) and for all r, p_r is the probability that B_r be optimal.

Sketch of the proof. The proof is adapted from the one of Dyer, Frieze and McDiarmid's inequality. For any feasible x, we can write

$$\begin{aligned} E[c^t x \mid B_r \text{ optimal }, c_{B_r}] &= c_{B_r}^t x_{B_r} + \textstyle\sum_{i \notin B_r} E[c_i \mid \ B_r \text{ optimal }, c_{B_r}]^t x_i \\ &= c_{B_r}^t x_{B_r} + \textstyle\sum_{i \notin B_r} (E[c_i \mid c_i \geq c_{B_r} A_{B_r}^{-1} a_i \geq 0, c_{B_r}]^t x_i \\ &\geq c_{B_r}^t x_{B_r} + \textstyle\sum_{i \notin B_r} E[c_i]^t x_i + \textstyle\sum_{i \notin B_r} c_{B_r} A_{B_r}^{-1} a_i x_i \\ &= c_{B_r}^t x_{B_r} + E[c_{N_r}]^t x_{N_r} + c_{B_r} A_{B_r}^{-1} A_{N_r} x_{N_r}. \end{aligned}$$

The second equality is where both the underlying mechanism of linear programming and Proposition 1 play a crucial rôle. Now since

$$A_{B_r} x_{B_r} + A_{N_r} x_{N_r} = Ax = b$$

and $E[z^* \mid \ B_r \text{ optimal }, c_{B_r}] = c_{B_r} A_{B_r}^{-1} b$, we get

$$E[c^t x \mid \ B_r \text{ optimal }, c_{B_r}] \geq E[c_{N_r}]^t x_{N_r} + E[z^* \mid \ B_r \text{ optimal }, c_{B_r}].$$

Now, taking the expectation relative to c_{B_r} and using the formula of total probabilities with respect to the events "B_r is optimal" for all r, we easily get the announced result. □

This theorem cannot be used in crude manner. It explicitely requires that all probabilities p_r be estimated which amounts to a usually hudge number in complex cases. The main idea for using this result is to restrict to the bases which appear most often in the experiments. The following bound appears more tractable in practice.

Corollary 2 *Let x be the solution of the linear program (3) where the costs are replaced by their expected values, let B be the optimal basis associated with x and let $\hat{p}_B$ be a lower bound to the probability that B be optimal basis in (1) with confidence level $1-\alpha$. Then with probability $1-\alpha$ we have*

$$\hat{p}_B E[c]^t x \leq E[z^*].$$

Remark 1 *Of course, one can add to $\hat{p}_B E[c]^t x$ several more terms of the form $\hat{p}_r \sum_{i \in B_r} E[c_i] x_i$ for as many other bases occuring with nonneglectable probability as we want, using underestimations $\hat{p}_r$ of p_r for each new r. This results in sharper lower bounds to the mean inspection time.*

4 Simulations results

In this section we present our simulation results. We generate a hundred different shortest path problems on the graph of Figure **??**. Each of these problems differs by the choice of the shape and scale parameters controlling the distribution of the cost on each edges. More precisely, for problem i, the cost c_j of edge j follows a Weibull distribution $W(\eta_j(i), \beta_j(i))$, where $\eta_j(i)$ was drawn at random from the normal distribution $\mathcal{N}(50, 100)$ and $\beta_j(i)$ was drawn from a uniform distribution over $[1, 2]$. For each problem, the expectation of the optimal cost is computed via Monte Carlo simulations over 1000 samples. Note that this is possible because of the simplicity of the chosen example. In order to compute our lower bound and following Remark 1, we use all the basis which occured to be optimal for at least one sample among the thousand samples. The bases are obtained as output of the simplex algorithm. Note that the only burden is to store these bases which is of course costless: we only take into account the small number of bases which frequently occur as optimal in the hundred tested samples of the shortest path problem. Finally, the underestimations $\hat{p}_r$ have been chosen as the lower bounds of the respective confidence intervals for the probabilities p_r with risk value $\alpha = 5\%$. The results are given in Figure 2 below.

In most cases, our lower bound is closer than the Dyer-Frieze-McDiarmid (DFM) upper bound to the expected cost in absolute value. Moreover, in general, the mean relative error between our bound and the expected value is less than 20% over the hundred generated problems (19.86 % in the displayed

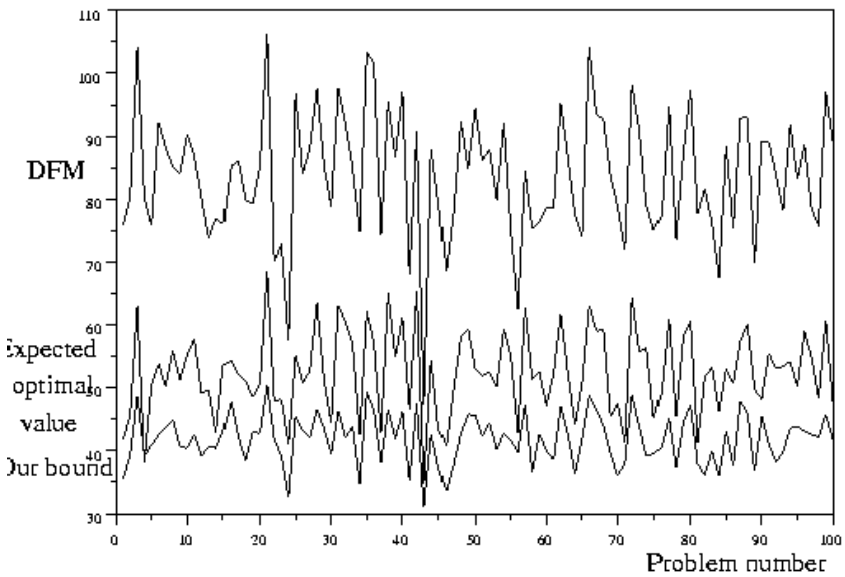

Figure 2: Comparison of the bounds for one hundred problems

example).

This very preliminary experimental results seem to be quite promising, and we plan to try our bound on real life problems in a short future.

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